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EXAMPLES OF ASYMPTOTIC ℓ_1 BANACH SPACES

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Abstract. Two examples of asymptotic ℓ_1 Banach spaces are given. The first, X_u, has an unconditional basis and is arbitrarily distortable. The second, X, does not contain any unconditional basic sequence. Both are spaces of the type of Tsirelson's.

Introduction

A Banach space $(X, \|\cdot\|)$ is λ-distortable ($\lambda > 1$) if there exists an equivalent norm $|\cdot|$ on X such that, for every infinite dimensional subspace Y of X,

$$\sup\left\{\frac{|y|}{|z|}:\ y, z \in Y,\ \|y\| = \|z\| = 1\right\} \geq \lambda.$$

X is abitrarily distortable if it is λ-distortable for every $\lambda > 1$.

The first example of an arbitrarily distortable Banach space was constructed by Th. Schlumprecht in [Schl]. Schlumprecht's space was the starting point for the construction by W.T. Gowers and B. Maurey of a Banach space not containing an unconditional basic sequence (u.b.s.) [G-M] and for the examples, due to W.T. Gowers, of a Banach space not containing ℓ_1, c_0 or a reflexive subspace [G1] and of a space without u.b.s. but with an asymptotically unconditional basis [G2].

A rapid development of the theory of Banach spaces followed the examples of Schlumprecht and Gowers-Maurey. We mention some results.

The notion of a hereditarily indecomposable Banach space was introduced in [G-M] and a new dichotomy property for Banach spaces regarding this notion was proved by Gowers [G3]. The remarkable answer to the distortion problem for ℓ_p by E. Odell and Th. Schlumprecht, namely the result that the spaces ℓ_p, $1 < p < \infty$, are arbitrarily distortable, also makes use of Schlumprecht's space. Finally, these results led to a new interest in the asymptotic structure of Banach spaces [Mi-To], [Ma-Mi-To].

It is well known (R.C. James, 1964, [J]) that ℓ_1 and c_0 are not distortable. On the other hand, it follows from a result of Milman ([Mi], 1971) and from [O-Schl] that a Banach space not containing c_0 or ℓ_1 contains a distortable subspace. However, the answer to the following question is still unknown. Does every Banach space contain either c_0 or ℓ_1 or an arbitrarily distortable subspace? It is proved in [Mi-To] that if X does not contain an arbitrarily distortable subspace then X has an asymptotic ℓ_p subspace (for some $1 \leq p < \infty$) or an asymptotic c_0 subspace. We recall the definition of this notion: *A Banach space with a normalized basis $\{e_k\}_{k=1}^{\infty}$ is asymptotic ℓ_p (resp. asymptotic c_0) if there exists a constant C such that for every n there*

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exists $N = N(n)$ *such that every sequence* $(x_i)_{i=1}^n$ *of successive normalized blocks of* $\{e_k\}_{k=1}^\infty$ *with* $N < \text{supp } x_1 < \text{supp } x_2 < \cdots < \text{supp } x_n$ *is* C*-equivalent to the canonical basis of* ℓ_p^n *(resp.* c_0^n*).* B. Maurey [Ma] has proved that an asymptotic ℓ_p space with an unconditinal basis which does not contain ℓ_1^n uniformly is arbitrarily distortable. In view of these results, a major class of spaces for which the distortion situation needs to be elucidated is the class of asymptotic ℓ_1 spaces. Note that it is unknown whether Tsirelson's space T contains an arbitrarily distortable subspace. So the following question was raised ([G2]).

Does there exist an asymptotic ℓ_1 arbitrarily distortable space?

In the first part of the present paper we give a positive answer to this question. In particular, we give an example of an arbitrarily distortable asymptotic ℓ_1 Banach space X_u with an unconditional basis.

Our construction has as starting point Tsirelson's celebrated example of the reflexive Banach space T not containing any ℓ_p . We recall, following T. Figiel and W. Johnson [F-J], the definition of Tsirelson's norm. Let $0 < \theta < 1$. On c_{00} (the space of finitely supported sequences) we define implicitly the norm $\|\cdot\|_T$ by

$$\|x\|_T = \max\left\{\|x\|_\infty, \sup\theta\sum_{i=1}^n \|E_i x\|_T\right\},$$

where the "sup" is taken over all families $\{E_1, E_2, \ldots, E_n\}$ of finite subsets of $\mathbf{N}$ such that $n \le E_1 < E_2 < \cdots < E_n$. Tsirelson's space is an asymptotic ℓ_1 space.

We consider the following generalization of Tsirelson's example. Let $\mathcal{M}$ be a family of finite subsets of $\mathbf{N}$ closed in the topology of pointwise convergence. A finite sequence $\{E_i\}_{i=1}^n$ of finite subsets of $\mathbf{N}$ is said to be $\mathcal{M}$-admissible if there exists a set $F = \{k_1, \ldots, k_n\} \in \mathcal{M}$ such that

$$k_1 \le E_1 < k_2 \le E_2 < \cdots < k_n \le E_n.$$

Let $0 < \theta < 1$. The Tsirelson type Banach space $T[\mathcal{M}, \theta]$ is the completion of c_{00} under the norm $\|\cdot\|_{\mathcal{M},\theta}$ which is defined by the following implicit equation:

$$\|x\|_{\mathcal{M},\theta} = \max\left\{\|x\|_\infty, \sup\theta\sum_{i=1}^n \|E_i x\|_{\mathcal{M},\theta}\right\},$$

where the "sup" is taken over all n and all $\mathcal{M}$-admissible sequences $\{E_i\}_{i=1}^n$. It is clear that Tsirelson's original space is $T[\mathcal{S}, \theta]$ where $\mathcal{S}$ is the Schreier family defined by

$$\mathcal{S} = \{F : F \subset \mathbf{N}, \#F \le \min F\}.$$

Consider $A_n = \{F : F \subset \mathbf{N}, \#F \le n\}$. S. Bellenot, [B], has proved the following result: For every $1 < p < \infty$ and $n \ge 2$ there exists $0 < \theta < 1$ such that $T[A_n, \theta]$ is isomorphic to ℓ_p. The spaces $T[\mathcal{F}_\xi, \theta]$ (the generalized Schreier families $\mathcal{F}_\xi$, $\xi < \omega_1$, introduced in [Al-Ar], are defined in 1(c) below) were introduced by the first named author in order to prove the following result: For every $\xi < \omega_1$ there exists a reflexive Banach space T_ξ such that every infinite dimensional subspace of T_ξ has Szlenk index greater than ξ (preprint, 1987). The general spaces $T[\mathcal{M}, \theta]$ were defined in [Ar-D].

The space X_u that we present here is defined using a "mixed Tsirelson's norm". Norms of this type are defined by sequences $\{\mathcal{M}_n\}_{n=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty$ such that each $\mathcal{M}_n$ is a family of finite subsets of $\mathbf{N}$ closed in the topology of pointwise convergence and $0<\theta_n<1$, $\lim_{n\to\infty}\theta_n=0$. The norm in the space $T[(\mathcal{M}_n,\theta_n)_{n=1}^{\infty}]$ is defined by

$$\|x\|=\max\left\{\|x\|_\infty,\ \sup_k\left\{\theta_k\sup\sum_{i=1}^{n}\|E_ix\|\right\}\right\},$$

where the inner "sup" is taken over all n and all $\mathcal{M}_k$-admissible families $(E_1,\dots,E_n)$. It is easy to see that if the Schreier family $\mathcal{S}$ is contained in one of the families $\mathcal{M}_n$ then the space $T[(\mathcal{M}_n,\theta_n)_{n=0}^{\infty}]$ is asymptotic ℓ^1. X_u is a space of the form $T[(\mathcal{M}_n,\theta_n)_{n=0}^{\infty}]$ where $(\mathcal{M}_n)_{n=1}^{\infty}$ is a subsequence of the sequence of the generalized Schreier families $(\mathcal{F}_n)_{n<\omega}$.

In the second part of the paper we give an example of an asymptotic ℓ_1 Banach space X which does not contain any unconditional basic sequence. In fact, X is hereditarily indecomposable. The question about the existence of such a space appears in [Ma] and [G2]. X is constructed via X_u in a way similar to the one used in [G-M] to pass from Schlumprecht's space to the Gowers-Maurey space. The basic idea for this comes from the fundamental construction by Maurey and Rosenthal [Ma-R] of a weakly null sequence without an unconditional basic subsequence.

Although our approach is different from that of Schlumprecht, Gowers and Maurey, it seems that the ingredients needed for the proofs are similar. So, for example, the semi-normalized (ϵ,j)-special convex combinations correspond to ℓ_N^1 vectors and the rapidly increasing (ϵ,j)-s.c.c.'s correspond to sums of rapidly increasing sequences.

1. Preliminaries

(a) Tsirelson type spaces

In [Ar-D] a space $T[\mathcal{M},\theta]$ has been defined, where $\mathcal{M}$ is a family of finite subsets of $\mathbf{N}$ closed in the topology of pointwise convergence and θ a real number with $0<\theta<1$.

We recall that definition. Given $\mathcal{M}$ as above, a family $(E_1,\dots,E_n)$ of succesive finite subsets of $\mathbf{N}$ is said to be $\mathcal{M}$-*admissible* if there exists a set $A=\{m_1,\dots,m_n\}\in\mathcal{M}$ such that $m_1\le E_1<m_2\le E_2<\cdots<m_n\le E_n$. The norm on the space $T[\mathcal{M},\theta]$ is defined implicitly by the formula

$$\|x\|=\max\{\|x\|_\infty,\theta\sup\sum_{i=1}^{n}\|E_ix\|\}$$

where the 'sup' is taken over all n and all $\mathcal{M}$-admissible $(E_1,\dots,E_n)$.

It is known that if the Cantor-Bendixson index of $\mathcal{M}$ is greater than ω, then the space $T[\mathcal{M},\theta]$ is reflexive. In 1.1 we prove a somewhat more general result.

(b) Mixed Tsirelson norms

Let $\{\mathcal{M}_k\}_{k=1}^{\infty}$ be families of finite subsets of $\mathbf{N}$ such that for each k:

(a) $\mathcal{M}_k$ is closed in the topology of pointwise convergence.
(b) $\mathcal{M}_k$ is adequate, i.e. if $A\in\mathcal{M}_k$ and $B\subset A$ then $B\in\mathcal{M}_k$.
(c) The Cantor-Bendixson index of $\mathcal{M}_k$ is greater than ω.

Let $\{\theta_k\}_{k=1}^{\infty}$ be a sequence of positive reals with each $\theta_k<1$ and $\lim\theta_k=0$. Then the mixed Tsirelson norm defined by $(\mathcal{M}_k,\theta_k)_{k=1}^{\infty}$ is given by the implicit relation

$$\|x\|=\max\left\{\|x\|_\infty,\sup_k\left\{\theta_k\sup\sum_{i=1}^{n}\|E_ix\|\right\}\right\},$$

where the inside "sup" is taken over all $\mathcal{M}_k$-admissible families $E_1, \dots, E_n$.

The Banach space defined by this norm is denoted by $T[(\mathcal{M}_k, \theta_k)_{k=1}^\infty]$.

1.1. Proposition. *The space $X = T[(\mathcal{M}_k, \theta_k)_{k=1}^\infty]$ is reflexive and $\{e_n\}_{n=1}^\infty$ is a 1-unconditional basis for X.*

Proof. The proof is similar to the original proof of Tsirelson in [T]. We first give an alternative definition of the norm of X.

We define inductively the following sets:

$K^0 = \{\pm e_n : n \in \mathbf{N}\}$.

Given K^s,

$$K^{s+1} = K^s \cup \Big\{ \theta_k(f_1 + \cdots + f_d\}) : k \in \mathbf{N}, d \in \mathbf{N}, f_i \in K^s, i = 1, \dots, d, \text{ supp } f_1 < \text{supp } f_2 < \cdots < \text{supp } f_d \text{ and the set } \{\text{supp } f_1, \dots, \text{supp } f_d\} \text{ is } \mathcal{M}_k\text{-admissible} \Big\}.$$

Finally, we set

$$K = \bigcup_{s=0}^\infty K^s.$$

Note that K is the smallest subset of B_{c_0} which contains $\pm e_n$ for all $n \in \mathbf{N}$ and has the property that $\theta_k(f_1 + \cdots + f_d)$ is in K whenever $f_1, \dots, f_d \in K$ and $\{\text{supp } f_1, \dots, \text{supp } f_d\}$ is $\mathcal{M}_k$-admissible.

For $x \in c_{00}$ we define

$$\|x\| = \sup_{f \in K} \langle x, f \rangle.$$

Then X is the completion of $(c_{00}, \|\cdot\|)$.

It is easy to see that $\{e_n\}_{n=1}^\infty$ is a 1-unconditional basis for X.

To show that X is reflexive, we have to show that the basis $\{e_n\}_{n=1}^\infty$ is shrinking and boundedly complete.

(a) $\{e_n\}_{n=1}^\infty$ is a shrinking basis for X.

Let $\theta = \max_k \theta_k < 1$. For $f \in X^*$ and $m \in \mathbf{N}$, denote by $Q_m(f)$ the restriction of f to the space generated by $\{e_k\}_{k \geq m}$. It suffices to prove the following: For every $f \in B_{X^*}$ there is $m \in \mathbf{N}$ such that $Q_m(f) \in \theta B_{X^*}$. Recall that $B_{X^*} = \overline{\text{co}(K)}$ where the closure is in the topology of pointwise convergence. We shall first prove the following:

Claim. For every $f \in \overline{K}$ there is m such that $Q_m(f) \in \theta\, \text{co}(\overline{K})$.

To prove this, let $f \in \overline{K}$ and let $\{f^n\}_{n=1}^\infty$ be a sequence in K converging pointwise to f.

If $f^n \in K^0$ for an infinite number of n, we have nothing to prove. So suppose that for every n there are $k_n \in \mathbf{N}$, a set $\{m_1^n, \dots, m_{d_n}^n\} \in \mathcal{M}_{k_n}$ and vectors $f_i^n \in K$, $i = 1, \dots, d_n$ such that $m_1^n \leq \text{supp } f_1^n < m_2^n \leq \text{supp } f_2^n < \cdots < m_{d_n}^n \leq \text{supp } f_{d_n}^n$ and $f^n = \theta_{k_n}(f_1^n + \cdots + f_{d_n}^n)$. If there is a subsequence of $\{\theta_{k_n}\}$ converging to 0, then $f = 0$. So we may suppose that there is a k such that $k_n = k$ for all n, i.e. $\theta_{k_n} = \theta_k$ and $\{m_1^n, \dots, m_{d_n}^n\} \in \mathcal{M}_k$.

Since $\mathcal{M}_k$ is compact, substituting $\{f^n\}$ with a subsequence we get that there is a set $\{m_1, \dots, m_d\} \in \mathcal{M}_k$ such that the sequence of indicator functions of the sets $\{m_1^n, \dots, m_{d_n}^n\}$ converges to the indicator function of $\{m_1, \dots, m_d\}$. So, for large n, $m_i^n = m_i$, $i = 1, \dots, d$, and $m_{d+1}^n \to \infty$ as $n \to \infty$.

Passing to a further subsequence of $(f^n)_{n=1}^\infty$, we get that there exist $f_i \in \overline{K}$, $i = 1, \dots, d$, with supp $f_i \subset [m_i, m_{i+1})$, $i = 1, \dots, d-1$, and supp $f_d \subset [m_d, \infty)$ such that $f_j^n \to f_j$ pointwise for $j = 1, \dots, d$. We conclude that $f = \theta_k(f_1 + \cdots + f_d)$, so $Q_{m_d}(f) = \theta_k f_d \in \theta\, \mathrm{co}(\overline{K})$.

The proof of the claim is complete. In particular we get that $\overline{K}$ is a weakly compact subset of c_0.

By standard arguments we can now pass to the case of $B_{X^*} = \overline{\mathrm{co}(K)}$.

(b) $\{e_n\}_{n=1}^\infty$ is a boundedly complete basis for X.

Suppose on the contrary that there exist $\epsilon > 0$ and a block sequence $\{x_i\}_{i=1}^\infty$ of $\{e_n\}_{n=1}^\infty$ such that $\sup_n \| \sum_{i=1}^n x_i \| \leq 1$ while $\|x_i\| \geq \epsilon$ for $i = 1, 2, \dots$.

Choose $n_0 \in \mathbf{N}$ such that $n_0 \theta_1 > \frac{1}{\epsilon}$. Using the fact that the $n_0 + 1$-derived set of $\mathcal{M}_1$ is non-empty, one can choose a set $\{m_1, \dots, m_{n_0}\} \in \mathcal{M}_1$ and a subset $\{x_{i_k}\}_{k=1}^{n_0}$ of $\{x_i\}_{i=1}^\infty$ such that

$$m_1 \leq \text{supp } x_{i_1} < m_2 \leq \text{supp } x_{i_2} < \cdots < m_{n_0} \leq \text{supp } x_{i_{n_0}}.$$

Then

$$\left\| \sum_{k=1}^{n_0} x_{i_k} \right\| \geq \theta_1 \sum_{k=1}^{n_0} \|x_{i_k}\| \geq n_0 \theta_1 \epsilon > 1,$$

a contradiction and the proof is complete.

(c) Generalized Schreier families

The *Schreier family* $\mathcal{S}$ is the set of all finite subsets of $\mathbf{N}$ satisfying the property $\#A \leq \min A$. It is easy to see that this family is closed in the topology of pointwise convergence.

1.2. Definition. Given $\mathcal{M}, \mathcal{N}$, families of finite subsets of $\mathbf{N}$ which are closed in the topology of pointwise convergence, the $\mathcal{M}$ *operation on* $\mathcal{N}$ is defined as

$$\mathcal{M}[\mathcal{N}] = \Big\{ F \subset \mathbf{N} : F = \bigcup_{i=1}^s F_i, \quad s \in \mathbf{N},\ F_i \in \mathcal{N},\ i = 1, \dots, s, \text{ and there exists a set } \{m_1, \dots, m_s\} \in \mathcal{M} \text{ such that } m_1 \leq F_1 < m_2 \leq F_2 < \cdots < m_s \leq F_s \Big\}.$$

$\mathcal{M}[\mathcal{N}]$ is a family of finite subsets of $\mathbf{N}$ which is closed in the topology of pointwise convergence.

1.3. Definition. The generalized Schreier families $\{\mathcal{F}_\xi\}_{\xi < \omega_1}$ are defined as follows:

$\mathcal{F}_0 = \{\{n\} : n \in \mathbf{N}\}$,

$\mathcal{F}_{\xi+1} = \mathcal{S}[\mathcal{F}_\xi]$.

For ξ a limit ordinal we let $\{\xi_n\}_{n=1}^\infty$ be a fixed sequence strictly increasing to ξ and set

$$\mathcal{F}_\xi = \{A \subset \mathbf{N} : n \leq \min A \text{ and } A \in \mathcal{F}_{\xi_n}\}.$$

The families $\{\mathcal{F}_\xi\}_{\xi < \omega_1}$ have been introduced in [Al-Ar].

Remark. It is easy to see that for $\xi_1, \xi_2 < \omega_1$ there exists $\xi < \omega_1$ such that $\mathcal{F}_{\xi_1}[\mathcal{F}_{\xi_2}] \subset \mathcal{F}_\xi$. In particular, for $m, n \in \mathbf{N}$, $\mathcal{F}_n[\mathcal{F}_m] = \mathcal{F}_{n+m}$.

In the sequel, we will only make use of the families $\mathcal{F}_n$, $n < \omega$.

We present now some properties of the families $\mathcal{F}_n$, $n = 1, 2, \dots$, which are important for our constructions.

1.4. Lemma. *Let $n \in \mathbf{N}$ and $F = \{s_1, s_2, \dots, s_d\} \subset \mathbf{N}$ with $s_1 < s_2 < \cdots < s_d$. Suppose that $F \in \mathcal{F}_n$. If $G = \{t_1, t_2, \dots, t_r\} \subset \mathbf{N}$ is such that $t_1 < t_2 < \cdots < t_r$, $r \le d$ and $s_p \le t_p$ for $p = 1, 2, \dots, r$, then $G \in \mathcal{F}_n$.*

This can be easily proved by induction on n.

1.5. Proposition. *Let $n \in \mathbf{N}$, $\epsilon > 0$. Denote by $|\cdot|_n$ the norm of the space $T[\mathcal{F}_n, \frac{1}{2}]$. There exists $m > n$ such that for every infinite subset D of $\mathbf{N}$ there exists a set $F \subset D$ with $F \in \mathcal{F}_m$ and a convex combination $x = \sum_{l \in F} a_l e_l$ with $\{a_l\}_{l \in F}$ decreasing and such that $|x|_n < \epsilon$.*

We first prove the following.

1.6. Lemma. *Let $t \ge 1$, $\epsilon > 0$, D be an infinite subset of $\mathbf{N}$. There exists a set $F \in \mathcal{F}_t$, $F \subset D$, and a convex combination $x = \sum_{l \in F} a_l e_l$ such that*
i) *$\{a_l\}_{l \in F}$ is in decreasing order,*
ii) *For every G in $\mathcal{F}_{t-1}$, $\sum_{l \in G} a_l < \epsilon$.*

Proof. The proof is by induction on t.

For $t = 1$, $\epsilon > 0$, we choose $n_0 > \frac{1}{\epsilon}$ and $F \subset D$ with $|F| = n_0$ and $n_0 \le F$. The vector $x = \frac{1}{n_0} \sum_{l \in F} e_l$ has the desired properties.

Suppose that we know the result for t. We prove it for $t+1$. Let $n_0 > \frac{2}{\varepsilon}$. Choose successively vectors x_k and integers n_k, $k = 1, \dots, n_0$, such that:

For $k = 1, \dots, n_0$, $n_k > 2n_{k-1}$ and x_k is a convex combination of the form $x_k = \sum_{l \in A_k} a_l e_l$, where
(a) $A_k \subset D \cap (n_{k-1}, n_k]$ and $A_k \in \mathcal{F}_t$,
(b) $\{a_l\}_{l \in A_k}$ is decreasing and, for $k \ge 2$, $\max_{l \in A_k} a_l < \min_{l \in A_{k-1}} a_l$,
(c) For every $B \in \mathcal{F}_{t-1}$ we have $\sum_{l \in B \cap A_k} a_l < \frac{1}{2n_{k-1}}$.

Set $F = \bigcup_{k=1}^{n_0} A_k$ and $x = \frac{1}{n_0} \sum_{k=1}^{n_0} x_k$. Then x has the desired properties. Indeed, $F \in \mathcal{F}_{t+1}$ and the coefficients are in decreasing order. To prove (ii) let $G \in \mathcal{F}_t$. Then $G = \bigcup_{i=1}^{s} G_i$ for some $s \in \mathbf{N}$ and sets G_i, $i = 1, \dots, s$, with $G_i \in \mathcal{F}_{t-1}$ and $s \le G_1 < G_2 < \cdots < G_s$. Let $k_0 \ge 1$ be such that $n_{k_0 - 1} < \min(G \cap F) \le n_{k_0}$. Then $s \le n_{k_0}$ and

$$\begin{aligned} \sum_{l \in G} a_l &= \sum_{l \in G \cap A_{k_0}} a_l + \sum_{k \ge k_0+1} \sum_{i=1}^{s} \sum_{l \in G_i \cap A_k} a_l \\ &< 1 + \sum_{k \ge k_0+1} \frac{s}{2n_{k-1}} \le 1 + \frac{n_{k_0}}{2} \sum_{k \ge k_0} \frac{1}{n_k} < 2. \end{aligned}$$

Thus, $\frac{1}{n_0} \sum_{l \in G} a_l < \epsilon$. This completes the proof.

Proof of Proposition 1.5. Choose l such that $\frac{1}{2^l} < \frac{\epsilon}{2}$ and set $m = ln + 1$. Using the previous lemma, choose a convex combination $x = \sum_{k \in F} a_k e_k$ such that $F \subset D$, $F \in \mathcal{F}_m$ and $\sum_{k \in G} a_k < \frac{\epsilon}{2}$ for every $G \in \mathcal{F}_{m-1}$. We claim that x is the desired vector. Indeed, let K be the norming set corresponding to the space $T[\mathcal{F}_n, \frac{1}{2}]$ as it is defined in the proof of Proposition 1.1. Let $\phi \in K$. Set $L = \left\{k \in \mathbf{N} : |\phi(k)| \ge \frac{1}{2^l}\right\}$. Then $L \in \mathcal{F}_{m-1}$. To see this, notice first that $\phi | L$ belongs to K^l, the set obtained at the l-th stage of the construction of K. Now, one can prove by induction on l that for every $f \in K^l$, $\operatorname{supp} f$ is in $\mathcal{F}_{l \cdot n} = \mathcal{F}_n[\cdots[\mathcal{F}_n]\cdots]$ (l-times). So $L \in \mathcal{F}_{l \cdot n} = \mathcal{F}_{m-1}$.

Therefore,

$$\begin{aligned}\Big|\phi(x)\Big| &\leq \Big|(\phi|L)(x)\Big| + \Big|(\phi|L^c)(x)\Big| \\ &\leq \sum_{k\in L} a_k + \frac{1}{2^l} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

The proof is complete.

2. The space X_u

We choose a sequence of integers $\{m_j\}_{j=0}^{\infty}$ such that $m_0 = 2$, and, for $j = 1, 2, \dots$, $m_j > m_{j-1}^{m_{j-1}}$. Inductively, using Proposition 1.5, we choose two subsequences $\{\mathcal{F}_{s_j}\}_{j=0}^{\infty}$ and $\{\mathcal{F}_{k_j}\}_{j=0}^{\infty}$ of $\{\mathcal{F}_n\}_{n=0}^{\infty}$ such that

(i) $k_0 = s_0 = 1$,

(ii) For every $n = 1, 2, \dots$, $\mathcal{F}_{s_n}$ has the following property:

For every infinite subset D of $\mathbf{N}$ there exist a set $F \in \mathcal{F}_{s_n}$ and a decreasing convex combination $x = \sum_{l\in F} a_l e_l$ such that $|x|_{k_{n-1}} < \frac{1}{m_{n+1}^4}$,

(iii) For every $n \geq 1$, $k_n = l_n \cdot (s_n + 1)$, where l_n is such that $2^{l_n} > m_n$. So, $\mathcal{F}_{k_n} = \mathcal{F}_{s_n+1}[\cdots[\mathcal{F}_{s_n+1}]\cdots]$ (l_n times).

We set $\mathcal{M}_j = \mathcal{F}_{k_j}$, $j = 0, 1, \dots$, and define X_u by

$$X_u = T\left[\left(\mathcal{M}_j, \frac{1}{m_j}\right)_{j=0}^{\infty}\right].$$

2.1. *Notation.* For $j = 0, 1, \dots$ we denote by $\|\cdot\|_j$ the norm of $T\left[\left(\mathcal{M}_n, \frac{1}{m_n}\right)_{n=0}^{j}\right]$ and by $\|\cdot\|_j^*$ the corresponding dual norm.

Remark. Notice that since $\mathcal{M}_n \subset \mathcal{F}_{k_j}$, $n = 0, \dots, j$, and $2 = m_0 < \cdots < m_j$, we have that, for every $x \in c_{00}$, $\|x\|_j \leq |x|_{k_j}$, where $|\cdot|_{k_j}$ is the norm of $T\left[\mathcal{F}_{k_j}, \frac{1}{2}\right]$.

2.2. Definition. Given $\epsilon > 0$ and $j = 0, 1, \dots$, an (ϵ, j)-*basic special convex combination* ((ϵ, j)-basic s.c.c.) is a vector of the form $\sum_{k\in F} a_k e_k$ such that $F \in \mathcal{M}_j$, $a_k \geq 0$, $\sum_{k\in F} a_k = 1$, $\{a_k\}_{k\in F}$ is decreasing and $\|\sum_{k\in F} a_k e_k\|_{j-1} < \epsilon$.

Remark. By the choice of s_j in the definition of X_u and the previous remark we get that for every j, every $\epsilon \geq \frac{1}{m_{j+1}^4}$ and every infinite subset D of $\mathbf{N}$, there exists an (ϵ, j)-basic s.c.c. of the form $\sum_{k\in F} a_k e_k$, where $F \in \mathcal{F}_{s_j}$ and $F \subset D$.

To fix notation, we repeat the definition of the set K of functionals that define the norm of the space X_u.

For $j = 0, 1, \dots$, we set $K_j^0 = \{\pm e_n : n \in \mathbf{N}\}$.

Assume that $\{K_i^n\}_{i=0}^{\infty}$ have been defined. Then we set

$$\begin{aligned}K^n &= \bigcup_{i=0}^{\infty} K_i^n \text{ and, for every } j, \\ K_j^{n+1} &= K_j^n \cup \left\{\frac{1}{m_j}(f_1 + \cdots + f_d) : \{\operatorname{supp} f_1 < \cdots < \operatorname{supp} f_d\} \text{ is } \mathcal{M}_j\text{-admissible and } f_1, \dots, f_d \text{ belong to } K^n\right\}.\end{aligned}$$

Set $K = \bigcup_{n=0}^{\infty} K^n$.

The norm $\|\cdot\|$ on X_u is

$$\|x\| = \sup\{f(x) : f \in K\}.$$

Notation. For $j = 0, 1, \dots$ we denote by A_j the set

$$A_j = \bigcup_{n=1}^{\infty} K_j^n.$$

That is, A_j consists of $\pm e_n$, $n \in \mathbf{N}$, and the elements $f \in K$ that are of the form

$$f = \frac{1}{m_j}(f_1 + \dots + f_d)$$

for some $d \in \mathbf{N}$ and some $\mathcal{M}_j$-admissible sequence $\{f_1, \dots, f_d\}$ of elements of K.

2.3. Definition. Let $m \in \mathbf{N}$, $\phi \in K^m \setminus K^{m-1}$. We call *analysis* of ϕ any sequence $\{K^s(\phi)\}_{s=0}^m$ of subsets of K such that:

1) For every s, $K^s(\phi)$ consists of successive elements of K^s and $\bigcup_{f \in K^s(\phi)} \operatorname{supp} f = \operatorname{supp} \phi$.
2) If f belongs to $K^{s+1}(\phi)$ then either $f \in K^s(\phi)$ or there exists j and $f_1, \dots, f_d \in K^s(\phi)$ with $\{\operatorname{supp} f_1, \dots, \operatorname{supp} f_d\}$ successive, $\mathcal{M}_j$-admissible and such that $f = \frac{1}{m_j}(f_1 + \dots + f_d)$.
3) $K^m(\phi) = \{\phi\}$.

Remark. Every $\phi \in K$ has an analysis. Also, if $f_1 \in K^s(\phi)$, $f_2 \in K^{s+1}(\phi)$, then either $\operatorname{supp} f_1 \subset \operatorname{supp} f_2$ or $\operatorname{supp} f_1 < \operatorname{supp} f_2$ or $\operatorname{supp} f_2 < \operatorname{supp} f_1$.

2.4. Definition. (a) Given $\phi \in K^m \setminus K^{m-1}$ and $\{K^s(\phi)\}_{s=0}^m$ a fixed analysis of ϕ, then for a given finite block sequence $\{x_k\}_{k=1}^l$ we set

$$s_k = \begin{cases} \max\{s : 0 \le s < m \text{ and there are at least two } f_1, f_2 \in K^s(\phi) \text{ such} \\ \qquad \text{that } \operatorname{supp} f_i \cap \operatorname{supp} x_k \neq \emptyset, i = 1, 2\}, \text{ if this set is non-empty,} \\ 0 \quad \text{if } \#(\operatorname{supp} x_k \cap \operatorname{supp} \phi) \le 1. \end{cases}$$

(b) For $k = 1, \dots, l$, we define the *initial* and *final part* of x_k with respect to $\{K^s(\phi)\}_{s=0}^m$, denoted by x_k' and x_k'' respectively, as follows: Let $\{f \in K^{s_k}(\phi) : \operatorname{supp} f \cap \operatorname{supp} x_k \neq \emptyset\} = \{f_1, \dots, f_d\}$, where $\operatorname{supp} f_1 < \dots < \operatorname{supp} f_d$. Then we set $x_k' = x_k | \operatorname{supp} f_1$, $x_k'' = x_k | \bigcup_{i=2}^d \operatorname{supp} f_i$.

A. Estimates on the basis $(e_n)_{n \in \mathbf{N}}$

2.5. Proposition. *For given $j \in \mathbf{N}$, $0 < \epsilon < \frac{1}{m_j^2}$ and $\sum_{k \in F} a_k e_k$ an (ϵ, j)-basic s.c.c. we have that: For $\phi \in K$*

$$\left| \phi\left(\sum_{k \in F} a_k e_k \right) \right| \le \begin{cases} \frac{1}{m_s} & \text{if } \phi \in A_s, s \ge j, \\ \frac{2}{m_s \cdot m_j} & \text{if } \phi \in A_s, s < j. \end{cases}$$

Proof. If $s \ge j$ then the estimate is obvious.

Assume that $s < j$ and for some $\phi \in K_s$, $|\phi(\sum a_k e_k)| > \frac{2}{m_s m_j}$. Without loss of generality we assume that $\phi(e_k) \ge 0$ for all k. Then $\phi = \frac{1}{m_s}(x_1^* + \dots + x_d^*)$, where $\{\operatorname{supp} x_1^* < \dots < \operatorname{supp} x_d^*\}$ is $\mathcal{M}_s$-admissible. We set

$$D = \left\{ k \in F : \sum_{i=1}^d x_i^*(e_k) > \frac{1}{m_j} \right\}.$$

Set $y_i^* = x_i^*|D$. Since each y_i^* has all its coordinates strictly greater than $\frac{1}{m_j}$, it is clear that y_i^* can be constructed from the set $K^0 = \{\pm e_n : n \in N\}$ using only operations of the form $\frac{1}{m_n}(f_1 + \cdots + f_d)$ where $\{\text{supp } f_l\}_{l=1}^d$ is $\mathcal{M}_n$-admissible and $n \leq j-1$. This means that, for each i, $\|y_i^*\|_{j-1}^* \leq 1$, so also $\left\|\frac{1}{m_s}(y_1^* + \cdots + y_d^*)\right\|_{j-1}^* \leq 1$. Then $\frac{1}{m_s}(y_1^* + \cdots + y_d^*)\left(\sum_{k\in F} a_k e_k\right) < \frac{1}{m_j^2}$. Hence,

$$\begin{aligned}\frac{1}{m_s}&(x_1^* + \cdots + x_d^*)(\sum_{k\in F} a_k e_k)\\ &= \frac{1}{m_s}(y_1^* + \cdots + y_d^*)(\sum_{k\in F} a_k e_k) + \frac{1}{m_s}(x_1^* + \cdots + x_d^*)(\sum_{k\in F\setminus D} a_k e_k)\\ &\leq \frac{1}{m_j^2} + \frac{1}{m_s m_j} < \frac{2}{m_s m_j},\end{aligned}$$

a contradiction and the proof is complete.

2.6. *Remark.* (a) It is easy to see that every (ϵ, j)-basic s.c.c. in X_u has norm greater than or equal to $\frac{1}{m_j}$. Therefore, for $\epsilon < \frac{1}{m_j^2}$, we get that the norm of the (ϵ, j)-basic s.c.c. is exactly $\frac{1}{m_j}$.

(b) It is crucial for the rest of the proof that for $s < j$, and $x_i^* \in K$, $i = 1, \ldots, d$, with $\{\text{supp } x_i^*\}_{i=1}^d$ $\mathcal{M}_s$-admissible,

$$\left|\frac{1}{m_s}(x_1^* + \cdots + x_d^*)\right|\left(\sum_{k\in F} a_k e_k\right) \leq \frac{2}{m_s m_j}.$$

In other words, for the normalized vector $m_j \sum_{k\in F} a_k e_k$ we have that

$$\left|\sum_{i=1}^d x_i^*\right|\left(m_j(\sum_{k\in F} a_k e_k)\right) \leq 2.$$

B. Estimates on block sequences

2.7. Definition. (a) Given a normalized block sequence $(x_k)_{k\in\mathbf{N}}$ in X_u, a convex combination $\sum_{i=1}^n a_i x_{k_i}$ is said to be an (ϵ, j)-special convex combination of $(x_k)_{k\in\mathbf{N}}$ ((ϵ, j)-s.c.c.) if there exist $p_1 < p_2 < \cdots < p_n$ such that $2 \leq \text{supp } x_{k_1} \leq p_1 < \text{supp } x_{k_2} \leq p_2 < \cdots < \text{supp } x_{k_n} \leq p_n$ and $\sum_{i=1}^n a_i e_{p_i}$ is an (ϵ, j)-basic s.c.c.

(b) An (ϵ, j)-s.c.c. is called *semi-normalized* if $\|\sum_{i=1}^n a_i x_{k_i}\| \geq \frac{1}{2}$.

Remark. Note that if $\sum_{i=1}^n a_i x_{k_i}$ is an (ϵ, j)-s.c.c., then the set $\{\text{supp } x_{k_1}, \ldots, \text{supp } x_{k_n}\}$ is not necessarily $\mathcal{M}_j$-admissible. On the other hand, the set $\{2, p_1, \ldots, p_{n-1}\}$ belongs to $S[\mathcal{M}_j]$, which gives in particular that $\|\sum_{i=1}^n a_i x_{k_i}\| \geq \frac{1}{2m_j}$.

The following lemma establishes the existence of semi-normalized $\left(\frac{1}{m_{j+1}^4}, j\right)$-s.c.c.'s in every block subspace of X_u.

2.8. Lemma. *Let $\{x_k\}_{k=1}^\infty$ be a normalized block sequence in X_u. Let $j \geq 1$ and $\epsilon > \frac{1}{m_{j+1}^4}$. Then there exists a finite block sequence $\{y_k\}_{k=1}^n$ of $\{x_k\}_{k=1}^\infty$ such that $\|y_k\| = 1$ and a convex combination $\sum_{k=1}^n a_k y_k$ is an (ϵ, j)-s.c.c. with $\|\sum_{k=1}^n a_k y_k\| > \frac{1}{2}$.*

Proof. Using the Remark following Definition 2.2, we choose a block sequence $\{y_k^1\}_{k=1}^\infty$ of $\{x_k\}_{k=1}^\infty$ such that each y_k^1 is an (ϵ, j)-s.c.c. of $\{x_k\}_{k=1}^\infty$ defined by an (ϵ, j)-basic s.c.c. z_k^1 such that supp $z_k^1 \in \mathcal{F}_{s_j}$, $k = 1, 2, \dots$. Note that for every k the set $\{\text{supp } x_i : \text{supp } x_i \subset \text{supp } y_k^1\}$ is $\mathcal{F}_{s_j+1}$-admissible.

If for some k_0, $\|y_{k_0}^1\| \geq \frac{1}{2}$ then we are done; if not we consider the normalized block sequence $x_k^1 = \frac{y_k^1}{\|y_k^1\|}$ and apply the same procedure for $\{x_k^1\}_{k=1}^\infty$ as we did for $\{x_k\}_{k=1}^\infty$. Thus we get a block sequence $\{y_k^2\}_{k=1}^\infty$ of $\{x_k^1\}_{k=1}^\infty$ such that each y_k^2 is an (ϵ, j)-s.c.c. of $\{x_k^1\}_{k=1}^\infty$ defined by a basic s.c.c. z_k^2 with supp $z_k^2 \in \mathcal{F}_{s_j}$. Note that for every k the set $\{\text{supp } x_i : \text{supp } x_i \subset \text{supp } y_k^2\}$ is $\mathcal{F}_{s_j+1}[\mathcal{F}_{s_j+1}]$-admissible (so $\mathcal{M}_j$-admissible). So, if there is no k such that $\|y_k^2\| \geq \frac{1}{2}$, then we get that $\frac{1}{m_j} \leq \left\|\frac{1}{2} y_k^2\right\| < \frac{1}{2^2}$, $k = 1, 2, \dots$.

Repeating the procedure l_j times, if we never get a y_k^i, $1 \leq i \leq l_j$, with $\|y_k^i\| \geq \frac{1}{2}$, then we arrive at a $y_k^{l_j}$ of the form $y_k^{l_j} = \sum_{i \in S} a_i t_i x_i$ where $\{\text{supp } x_i\}_{i \in S}$ is $\mathcal{F}_{l_j(s_j+1)}$-admissilble (that is, $\mathcal{M}_j$-admissible), $\sum_{i \in S} a_i = 1$ and $t_i \geq 2^{l_j - 1}$, for all $i \in S$. Then

$$\frac{1}{m_j} \leq \frac{1}{2^{l_j-1}} \left\| y_k^{l_j} \right\| < \frac{1}{2^{l_j}},$$

a contradiction since $m_j < 2^{l_j}$.

2.9. Proposition. *Let $j \in \mathbf{N}$. Let $\{x_k\}_{k=1}^n$ be a finite block sequence of normalized vectors in X_u. Let $\{p_1, \dots, p_n\}$ be such that* supp $x_{k_1} \leq p_1 <$ supp $x_{k_2} \leq p_2 < \cdots <$ supp $x_n \leq p_n$ *and suppose that $\{p_1, \dots, p_n\} \in \mathcal{M}_j$. Then, for every $r \leq j$ and every $\phi \in A_r$, there exists $\psi \in \text{co}(A_r)$ such that $|\phi(x_k)| \leq 2\psi(m_j e_{p_k})$, $k = 1, \dots, n$.*

Proof. Let $r \leq j$ and $\phi \in A_r$. Assume that $\phi \in K^m \setminus K^{m-1}$ for some $m \geq 0$ and let $\{K^s(\phi)\}_{s=0}^m$ be an analysis of ϕ. Let x_k', x_k'' be the initial and final part of x_k with respect to $\{K^s(\phi)\}_{s=0}^m$.

We shall define $\psi', \psi'' \in A_r$ such that for each k, $|\phi(x_k')| \leq \psi'(m_j e_{p_k})$ and $|\phi(x_k'')| \leq \psi''(m_j e_{p_k})$.

Construction of ψ'. For $f \in \bigcup_{s=0}^m K^s(\phi)$, we set

$$D_f = \{k : \text{supp } \phi \cap \text{supp } x_k' = \text{supp } f \cap \text{supp } x_k'\}.$$

By induction on $s = 0, \dots, m$, we shall define for every $f \in \bigcup_{s=0}^m K^s(\phi)$ a function g_f with the following properties:

(a) g_f is supported on $\{p_k : k \in D_f\}$.

(b) For $k \in D_f$, $|f(x_k')| \leq m_j g_f(e_{p_k})$.

(c) $g_f \in K$. Moreover, if $q \leq j$ and $f \in A_q$, then $g_f \in A_q$.

For $s = 0$, $f = \pm e_m^* \in K^0(\phi)$, $D_f \neq \emptyset$ only if for some k, $x_k' |\text{supp } \phi = \lambda e_m$, $|\lambda| \leq 1$. We then set $g_f = e_{p_k}^*$.

Let $s > 0$. Suppose that g_f have been defined for all $f \in \bigcup_{t=0}^{s-1} K^t(\phi)$. Let $f = \frac{1}{m_q}(f_1 + \cdots + f_d) = K^s(\phi) \setminus K^{s-1}(\phi)$, where $f_i \in K^{s-1}(\phi)$, $i = 1, \dots, d$, and $\{\text{supp } f_1, \dots, \text{supp } f_d\}$ is $\mathcal{M}_q$-admissible.

Let $I = \{i : 1 \leq i \leq d, D_{f_i} \neq \emptyset\}$.

Let $T = D_f \setminus \bigcup_{i \in I} D_{f_i}$.

Suppose first that $q \leq j$. We set

$$g_f = \frac{1}{m_q} \left(\sum_{i \in I} g_{f_i} + \sum_{k \in T} e_{p_k}^* \right).$$

Property (a) is obvious. For (b) we have:

If $k \in D_{f_i}$ for some $i \in I$,

$$|f(x'_k)| = \frac{1}{m_q}|f_i(x'_k)| \leq \frac{1}{m_q} g_{f_i}(m_j e_{p_k}) = g_f(m_j e_{p_k}),$$

using the inductive hypothesis.

For $k \in T$ we get

$$|f(x'_k)| = \frac{1}{m_q}\left|\sum_{i=1}^{d} f_i(x'_k)\right| \leq 1 \leq \frac{m_j}{m_q} = \frac{1}{m_q} e^*_{p_k}(m_j e_{p_k}) = g_f(m_j e_{p_k}).$$

To show that $g_f \in A_q$, we need to show that the set $\{\operatorname{supp} g_{f_i} : i \in I\} \cup \{\{p_k\} : k \in T\}$ is $\mathcal{M}_q$-admissible.

Let $G = \{t_1 < t_2 < \cdots < t_r\}$ be an ordering of the set $\{p_k : k \in T\} \cup \{\min\{p_k : k \in D_{f_i}\}, i \in I\}$. Set $F = \{s_1, s_2, \cdots, s_d\}$ where $s_i = \min(\operatorname{supp} f_i)$, $i = 1, \dots, d$. Then $F \in \mathcal{M}_q$. By the definition of x'_k, if $k \in T$ there is $f_i \in \{1, \dots, d\} \setminus I$ such that $\operatorname{supp} f_i \cap \operatorname{supp} x'_k \neq \emptyset$, $\operatorname{supp} f_i \cap \operatorname{supp} x'_m = \emptyset$ for all $m \neq k$. This shows that $r \leq d$ and $s_l \leq t_l$ for all $l \leq r$. Hence, by Lemma 1.4, $G \in \mathcal{M}_q$.

Suppose now that $q > j$. Then we set $g_f = \frac{1}{m_j}\left(\sum_{i \in I} g_{f_i} + \sum_{k \in T} e^*_{p_k}\right)$. Since $\{p_1, \dots, p_k\} \in \mathcal{M}_j$, it is obvious that $g_f \in K$.

Properties (a) and (b) are also easily checked.

The construction of ψ'' is similar.

Finally, we set $\psi = \frac{1}{2}(\psi' + \psi'')$.

2.10. Corollary. *Let $j \in \mathbf{N}$, $0 < \epsilon < \frac{1}{m_j^2}$. Let $\sum_{k=1}^{n} a_k x_k$ be an (ϵ, j)-s.c.c. Then, for $q < j$, $\phi \in A_q$, $|\phi(\sum a_k x_k)| \leq \frac{4}{m_q}$.*

Proof. Combine Propositions 2.5 and 2.9.

2.11. Definition. For $j = 1, 2, \dots$, $\epsilon > 0$, a finite block sequence $\{y_k\}_{k=1}^{n}$ is said to be an (ϵ, j)-*rapidly increasing sequence* if the following are satisfied:

(a) There exist $\{a_k\}_{k=1}^{n}$ with $a_k \geq 0$, $\sum a_k = 1$ such that $\sum_{k=1}^{n} a_k y_k$ is an (ϵ, j)-s.c.c.

(b) There exist $j_1, \dots, j_n$ such that:

(i) $j + 2 < 2j_1 < \cdots < 2j_n$,

(ii) each y_k is a semi-normalized $\left(\frac{1}{m^4_{2j_k}}, 2j_k\right)$-s.c.c.

(iii) the ℓ_1-norm of y_k is dominated by $\frac{m_{2j_{k+1}}}{m_{2j_{k+1}-1}}$.

The convex combination $y = \sum_{k=1}^{n} a_k y_k$, where $\{a_k\}_{k=1}^{n}$ is as in (a), is said to be an (ϵ, j)-*rapidly increasing s.c.c.*

2.12. Proposition. *Let $j \geq 1$. Let $\{y_k\}_{k=1}^{n}$ be an (ϵ, j)-rapidly increasing sequence and $(p_i)_{i=1}^{n}$ be such that* $\operatorname{supp} y_1 \leq p_1 < \operatorname{supp} y_2 \leq p_2 < \cdots \leq p_{n-1} < \operatorname{supp} y_n \leq p_n$ *and $\{p_1, \dots, p_n\} \in \mathcal{M}_j$. Let j_k be as in Definition 2.11. Then, for every $\phi \in A_r$ there exists $\psi \in co(K)$, such that for $k = 1, \dots, n$,*

$|\phi(y_k)| \leq 8\psi(e_{p_k})$. Moreover,

if $r < 2j_1$ then $\psi \in \operatorname{co} A_r$,

if $2j_1 \leq r \leq 2j_n$ then ψ is of the form $\psi = \frac{1}{2}\psi_1 + \frac{1}{2}e_{p_k}$, where $\psi_1 \in \operatorname{co}(A_{r-1})$, $p_k \notin \operatorname{supp} \psi_1$ and k is such that $2j_k \leq r < 2j_{k+1}$.

Proof. The construction is similar to the one in the proof of Proposition 2.9.

Let $\phi \in A_r$. Assume that $\phi \in K^m \setminus K^{m-1}$ and let $\{K^s(\phi)\}_{s=0}^m$ be an analysis of ϕ. Let y_k' and y_k'' be the initial and final part of y_k with respect to $\{K^s(\phi)\}_{s=0}^m$.

We shall define ψ' and ψ'' so that $|\phi(y_k')| \le 4\psi'(e_{p_k})$ and $|\phi(y_k'')| \le 4\psi''(e_{p_k})$.

Construction of ψ'. For $f \in \bigcup_{s=0}^m K^s(\phi)$, we set

$$D_f = \{k : \text{supp } \phi \cap \text{supp } y_k' = \text{supp } f \cap \text{supp } y_k' \neq \emptyset\}.$$

By induction on $s = 0, \dots, m$, we shall define for every $f \in \bigcup_{s=0}^m K^s(\phi)$ a function g_f with the following properties:

a) g_f is supported on $\{p_k : k \in D_f\}$,
b) $|f(y_k')| \le 4g_f(e_{p_k})$ for $k \in D_f$.
c) $g_f \in K$. Moreover, $g_f \in A_q$, if $q < 2j_1$ and $g_f = \frac{1}{2}g_f' + \frac{1}{2}e_{p_k}$, with $g_f' \in A_{q-1}$, $p_k \notin \text{supp } g_f'$, if $2j_k \le q < 2j_{k+1}$.

Let $s > 0$. Suppose that g_f have been defined for all $f \in \bigcup_{t=0}^{s-1} K^t(\phi)$. Let $f = \frac{1}{m_q}(f_1 + \cdots + f_d) \in K^s(\phi) \setminus K^{s-1}(\phi)$,

Case 1. $q < 2j_1$.

Let $I = \{i : 1 \le i \le d, D_{f_i} \neq \emptyset\}$ and $T = D_f \setminus \bigcup_{i\in I} D_{f_i}$. We set

$$g_f = \frac{1}{m_q}\left(\sum_{i\in I} g_{f_i} + \sum_{k\in T} e_{p_k}^*\right).$$

Properties (a) and (b) for the case $k \in \bigcup_{i\in I} D_{f_i}$ follow easily from the inductive assumption. For $k \in T$ we get

$$|f(y_k)| = \frac{1}{m_q}\left|\sum f_i(y_k)\right| \le \frac{4}{m_q} \le 4g_f(e_{p_k}),$$

by Corollary 2.10, since $q < 2j_k$ for all k.

The proof that $g_f \in A_q$ is as in the proof of Proposition 2.9 (Case $q < j$).

Case 2. $q \ge 2j_1$.

Let $1 \le t \le n$ be such that $2j_t \le q < 2j_{t+1}$. If $t \notin D_f$ or $t \in \bigcup_{i\in I} D_{f_i}$ then we set

$$g_f = \frac{1}{m_{q-1}}\left(\sum_{i\in I} g_{f_i} + \sum_{k\in T} e_{p_k}^*\right).$$

Then, clearly, $g_f \in A_{q-1}$. For $k \in D_{f_i}$ for some $i \in I$, $|f(y_k')| = \frac{1}{m_q}|f_i(y_k')| \le \frac{4}{m_q}(e_{p_k}) \le \frac{1}{m_{q-1}}g_{f_i}(e_{p_k}) = g_f(e_{p_k})$. For $k \in T$, if $k < t$ then $2j_{k+1} \le 2j_t \le q$, so we get

$$\begin{aligned} |f(y_k')| &= \frac{1}{m_q}\left|\left(\sum_{i=1}^d f_i\right)(y_k')\right| \le \frac{1}{m_q}\|y_k'\|_{\ell_1} \le \frac{1}{m_q}\frac{m_{2j_{k+1}}}{m_{2j_{k+1}-1}} \\ &\le \frac{1}{m_{q-1}} = \frac{1}{m_{q-1}}e_{p_k}^*(e_{p_k}) = g_f(e_{p_k}). \end{aligned}$$

If $k \in T$ and $k > t$ then $q < 2j_k$, so by Corollary 2.10 we get

$$|f(y_k')| = \frac{1}{m_q}\left|\left(\sum_{i=1}^d f_i\right)(y_k)\right| \le \frac{4}{m_q} \le \frac{1}{m_{q-1}} = g_f(e_{p_k}).$$

Now if $t \in D_f \setminus \bigcup_{i\in I} D_{f_i}$ then we set

$$g_f = \frac{1}{2}\left[\frac{1}{m_{q-1}}\left(\sum_{i\in I} g_{f_i} + \sum_{\substack{k\in T\\ k\neq t}} e^*_{p_k}\right)\right] + \frac{1}{2}e^*_{p_t}.$$

Then $g_f \in K$. In particular, $2g_f|\{e_{p_k} : k \in D_f, k \neq t\} \in A_{q-1}$. In the same manner as before one can check that for every $k \in D_f$

$$|f(y'_k)| \leq 4g_f(e_{p_k}).$$

This completes the proof for ψ'.

The construction of ψ'' is similar.

Finally, we set $\psi = \frac{1}{2}(\psi' + \psi'')$.

2.13. Proposition. *Let* $0 < \epsilon < \frac{1}{m_j^2}$ *and* $\sum_{k=1}^n a_k y_k$ *be an* (ϵ, j)*-rapidly increasing s.c.c. Then for* $i = 0, 1, 2, \ldots$, ϕ *in* A_i*, we have the following estimates:*

(a) $|\phi(\sum_{k=1}^n a_k y_k)| \leq \frac{16}{m_i m_j}$, *if* $i < j$,
(b) $|\phi(\sum_{k=1}^n a_k y_k)| \leq \frac{8}{m_i}$, *if* $j \leq i < 2j_1$,
(c) $|\phi(\sum_{k=1}^n a_k y_k)| \leq \frac{4}{m_{i-1}} + 4|a_{k_0}|$, *if* $2j_{k_0} \leq i < 2j_{k_0+1}$.

Proof. It follows easily from Propositions 2.5 and 2.12.

2.14. Corollary. *If* $\sum_{k=1}^n a_k y_k$ *is a* $\left(\frac{1}{m_j^2}, j\right)$*-rapidly increasing s.c.c. then*

$$\frac{1}{4m_j} \leq \left\|\sum_{k=1}^n a_k y_k\right\| \leq \frac{8}{m_j}.$$

2.15. Corollary. X_u *is arbitrarily distortable.*

Proof. Choose i_0 arbitrarily large. Let

$$|||x||| = \frac{1}{m_{i_0}}\|x\| + \sup\{\phi(x) : \phi \in A_{i_0}\}.$$

Let Y be a block subspace of X_u. Let $j > i_0$. Using Lemma 2.8, we can choose the following vectors in Y

$$\begin{aligned} y &= \sum_{k=1}^n a_k y_k, \text{ a } \left(\frac{1}{m_j^2}, j\right)\text{-rapidly increasing s.c.c.} \\ z &= \sum_{l=1}^m b_l z_l, \text{ a } \left(\frac{1}{m_{i_0}^2}, i_0\right)\text{-rapidly increasing s.c.c.} \end{aligned}$$

Then, by Proposition 2.13 and Corollary 2.14,

$$|||m_j y||| \leq \frac{8}{m_{i_0}} + \frac{16}{m_{i_0}} = \frac{24}{m_{i_0}} \text{ while } \|m_j y\| \geq \frac{1}{4},$$

$$|||m_{i_0} z||| \geq \frac{1}{4} \text{ while } \|m_{i_0} z\| \leq 8.$$

This completes the proof.

3. The space X

We turn now to defining the Banach space X not containing any unconditional basic sequence. The norm of the space is related to that of X_u introduced in the previous section. Specifically, the norm will be defined by a family $\{B_j\}_{j=0}^{\infty}$ of subsets of c_{00} such that each B_j is contained in the set A_j used in the definition of X_u. Let K be the norming set for X_u defined in Section 2. Note that K is countable. We consider the set

$$G = \{(x_1^*, x_2^*, \dots, x_k^*) : k \in \mathbf{N},\ x_i^* \in K, i = 1, \dots, k \text{ and } x_1^* < x_2^* < \dots < x_k^*\}\,.$$

Since G is countable, there exists a one to one function $\Phi : G \longrightarrow \{2j\}_{j=0}^{\infty}$ with the following property:

For every $(x_1^*, \dots, x_k^*) \in G$, let j_1 be minimal such that $x_1^* \in A_{j_1}$ and $j_l = \Phi(x_1^*, \dots, x_{l-1}^*)$, $l = 2, \dots, k$. Then $j_l > j_{l-1},\ l = 2, \dots, k$.

For $n = 0, 1, 2, \dots$ we define by induction sets $\{L_j^n\}_{j=0}^{\infty}$ such that L_j^n is a subset of K_j^n and $\{L_j^n\}_{n=0}^{\infty}$ is an increasing family.

For $j = 0, 1, \dots$ we set

$$L_j^0 = \{\pm e_n : n = 1, 2, \dots\}.$$

Suppose that $\{L_j^n\}_{j=0}^{\infty}$ have been defined and set for every j

$$L_{2j}^{n+1} = L_{2j}^n \cup \left\{ \frac{1}{m_{2j}} (x_1^* + \dots + x_d^*) : d \in \mathbf{N},\ x_i^* \in \bigcup_{t=0}^{\infty} L_t^n, \right.$$
$$\left. (\text{supp } x_1^*, \dots, \text{supp } x_d^*) \ \text{ is } \mathcal{M}_{2j} \text{-admissible} \right\},$$

$$L'^{\,n+1}_{2j+1} = L_{2j+1}^n \cup \left\{ \frac{1}{m_{2j+1}} (x_1^* + \dots + x_d^*) : d \in \mathbf{N},\ x_1^* \in L_{2k}^n \text{ for some } \right.$$
$$k > 2j+1,\ x_i^* \in L^n_{\Phi(x_1^*, \dots, x_{i-1}^*)} \text{ for } 1 < i \le d$$
$$\left. \text{and } (\text{supp } x_1^*, \dots, \text{supp } x_d^*) \text{ is } \mathcal{M}_{2j+1}\text{-admissible} \right\}$$

and

$$L_{2j+1}^{n+1} = \left\{ \pm E_s x^* : x^* \in L'^{\,n+1}_{2j+1}, s \in \mathbf{N}, E_s = \{s, s+1, \dots\} \right\}.$$

This completes the definition of L_j^n, $n = 0, 1, 2, \dots$, $j = 0, 1, 2, \dots$. It is obvious that each L_j^n is a subset of the corresponding set K_j^n.

We set $B_j = \bigcup_{n=1}^{\infty} L_j^n$ and we consider the norm on c_{00} defined by the family $L = \bigcup_{j=0}^{\infty} B_j$. The space X is the completion of c_{00} under this norm. It is easy to see that $\{e_n\}_{n=1}^{\infty}$ is a bimonotone basis for X.

3.1. *Remark.* An alternative implicit definition of the norm of the space X is the following. For $x \in c_{00}$,

$$\|x\| = \max\left\{\|x\|_\infty, \sup\left\{\frac{1}{m_{2j}}\sum_{k=1}^{n}\|E_k x\|, j \in \mathbf{N}, n \in \mathbf{N}, \{E_1 < \cdots < E_n\} \text{ is } \mathcal{M}_{2j}\text{-admissible}\right\}, \sup\left\{|\phi(x)| : \phi \in \bigcup_{j=0}^{\infty} B^{2j+1}\right\}\right\}.$$

Hence, for $j = 0, 1, 2, \ldots$ and for $x_1 < x_2 < \cdots < x_n$ in c_{00} such that $\{\text{supp } x_1, \text{supp } x_2, \ldots, \text{supp } x_n\}$ is $\mathcal{M}_{2j}$-admissible, we have that $\|\sum_{k=1}^{n} x_k\| \geq \frac{1}{m_{2j}}\sum_{k=1}^{n}\|x_k\|$. In particular, setting $j = 0$ we get that X is an asymptotic-ℓ_1 space.

For $\epsilon > 0$, $j = 1, 2, \ldots$, (ϵ, j)-special convex combinations and (ϵ, j)-rapidly increasing sequences are defined in X exactly as in X_u.

Using the above remark, one can prove the following result in the same manner as Lemma 2.8.

3.2. Lemma. *For $j = 1, 2, \ldots$ and every normalized block sequence $\{x_k\}_{k=1}^{\infty}$ in X there exists a finite block sequence $\{y_s\}_{s=1}^{n}$ of $\{x_k\}_{k=1}^{\infty}$ such that $\sum_{s=1}^{n} a_s y_s$ is a semi-normalized $\left(\frac{1}{m_{2j}^4}, 2j\right)$-s.c.c.*

3.3. Proposition. *Let $\sum_{k=1}^{r} a_k x_k$ be a $\left(\frac{1}{m_j^2}, j\right)$-s.c.c. defined by an $\left(\frac{1}{m_j^2}, j\right)$-basic s.c.c. $\sum_{k=1}^{n} a_k e_{p_k}$. Then for every $s \leq j$ and ϕ in B_s there exists ψ in A_s such that*

$$\left|\phi\left(\sum_{k=1}^{n} a_k x_k\right)\right| \leq 2\psi\left(m_j \sum_{k=1}^{n} a_k e_{p_k}\right).$$

The proof of this is similar to the proof of Proposition 2.9.

3.4. Proposition. *Let $\sum_{k=1}^{n} b_k x_k$ be a $\left(\frac{1}{m_j^4}, j\right)$-rapidly increasing s.c.c. in X. Then for $i \in \mathbf{N}$, $\phi \in B_i$, we have the following estimates:*

(a) $|\phi(\sum_{n=1}^{k} b_k x_k)| \leq \frac{16}{m_i m_j}$ *if $i < j$,*

(b) $|\phi(\sum_{n=1}^{k} b_k x_k)| \leq \frac{8}{m_i}$ *if $j \leq i < 2j_1$,*

(c) $|\phi(\sum_{n=1}^{k} b_k x_k)| \leq \frac{4}{m_{i-1}} + 4|b_{k_0}|$ *if $2j_{k_0} \leq i < 2j_{k_0+1}$.*

In particular, $\|\sum_{k=1}^{n} b_k x_k\| \leq \frac{8}{m_j}$.

This is proved similarly to Proposition 2.13.

The following proposition is the main result of this section.

3.5. Proposition. *Let $j > 100$ and suppose that $\{j_k\}_{k=1}^{n}$, $\{y_k^*\}_{k=1}^{n}$ and $\{\theta_k\}_{k=1}^{n}$ are such that*

(i) *Each y_k is a $\left(\frac{1}{m_{2j_k}^4}, 2j_k\right)$-rapidly increasing s.c.c. in X, the sequence $\{\text{supp } y_k\}_{k=1}^{n}$ is $\mathcal{F}_{s_{2j+1}}$-admissible and there exists a decreasing sequence $\{a_k\}_{k=1}^{n}$ such that $\sum_{k=1}^{n} a_k y_k$ is a $\left(\frac{1}{m_{2j+2}^4}, 2j+1\right)$-s.c.c.*

(ii) $y_k^* \in L_{2j_k}$, $y_k^*(y_k) \geq \frac{1}{4m_{2j_k}}$ *and* supp $y_k^* \subset [\min\text{supp } y_k, \max\text{supp } y_k]$.

(iii) $\frac{1}{8} \le \theta_k \le 4$ *and* $y_k^*(m_{2j_k}\theta_k y_k) = 1$.
(iv) $j_1 > 2j+1$ *and* $2j_k = \Phi\left(y_1^*, \dots, y_{k-1}^*\right)$, $k = 2, \dots, n$.

Let $(\epsilon_k)_{k=1}^n$ *be such that* $\epsilon_k = 1$ *if* k *is even and* $\epsilon_k = -1$ *if* k *is odd. Then*

$$\|\sum_{k=1}^n \epsilon_k a_k m_{2j_k}\theta_k y_k\| \le \frac{100}{m_{2j+2}}.$$

Note that for $\{y_k\}_{k=1}^n$, $\{y_k^*\}_{k=1}^n$, $\{a_k\}_{k=1}^n$, $\{\theta_k\}_{k=1}^n$ satisfying the assumptions of Proposition 3.5 we have that the functional $\psi = \frac{1}{m_{2j+1}}(\sum_{k=1}^n y_k^*)$ belongs to B_{2j+1}, so

$$\|\sum_{k=1}^n a_k m_{2j_k}\theta_k y_k\| \ge \frac{1}{m_{2j+1}}.$$

Thus, the fact that X does not contain an unconditional basic sequence will follow from Proposition 3.5, provided that we show the following:

For all $j > 100$, every block subspace Y of X contains a sequence $\{y_k\}_{k=1}^n$, satisfying the assumptions of Proposition 3.5.

Indeed, in Proposition 3.12 we show that, for arbitrary block subspaces U, V of X, the vectors y_k, $k = 1, \dots, n$, can be chosen to belong alternately to U and V; this implies that X is actually Hereditarily Indecomposable.

The proof of Proposition 3.5 is given in several steps. Our aim is to show that, for every $\phi \in \bigcup_{i=0}^\infty B_i$,

$$\phi\left(\sum \epsilon_k a_k m_{2j_k}\theta_k y_k\right) \le \frac{100}{m_{2j+2}}.$$

The cases $\phi \in B_{2j+1}$, $\phi \in \bigcup_{i\ge 2j+2} B_i$ and $\phi \in \bigcup_{i\le 2j} B_i$ are considered separately, in Propositions 3.9, 3.10 and 3.11 respectively.

3.6. Lemma. *Let* $\sum_{k=1}^n a_k x_k$ *be an* (ϵ, j)*-s.c.c. and* $i < j$. *Suppose that* $z_l^* \in L$, $l = 1, \dots, d$, *and* $\{\text{supp } z_l^*\}_{l=1}^d$ *is* $\mathcal{M}_i$*-admissible. Let* $\{k_t\}_{t=1}^s$ *be a subset of* $\{1, \dots, n\}$ *with the following property: There exists a one-to-one correspondence* $x_{k_t} \to z_{l_t}^*$, *such that* $z_{l_t}^*(x_{k_t}) \ne 0$, $t = 1, \dots, s$. *Then* $\sum_{t=1}^s a_{k_t} < m_i \cdot \epsilon$.

Proof. Let $\sum_{k=1}^n a_k e_{p_k}$ be the (ϵ, j)-basic s.c.c. that defines the s.c.c. $\sum_{k=1}^n a_k x_k$. It is easy to check that the set $(e_{p_{k_t}})_{t=1}^s$ is $\mathcal{M}_i$-admissible, hence

$$\frac{1}{m_i}\sum_{t=1}^s a_{k_t} \le \left\|\sum_{t=1}^s a_{k_t} e_{p_{k_t}}\right\|_i < \epsilon$$

and the proof is complete.

3.7. Lemma. *Let* y *be a* $\left(\frac{1}{m_{2j}^4}, 2j\right)$*-rapidly increasing s.c.c. and* $z_1^*, \dots, z_d^*$ *be in* $B_{2t_1}, \dots, B_{2t_d}$, *respectively, such that* $t_k \ne j$ *for all* $k = 1, \dots, d$, $\text{supp } z_1^* < \dots < \text{supp } z_d^*$ *and* $\frac{1}{m_i}\sum_{k=1}^d z_k^* \in B_i$ *for some* $i < 2j$. *Then*

$$|(z_1^* + \dots + z_d^*)(y)| < \sum_{k=1}^{d_1} \frac{16}{m_{2t_k} m_{2j}} + \sum_{k=d_1+1}^{d} \frac{8}{m_{2t_k - 1}} + \frac{1}{m_{2j}^2},$$

where $t_1 < \dots < t_{d_1} < j < t_{d_1+1} < \dots < t_d$.

Proof. Let $y=\sum_{n=1}^{l} a_n x_n$ be the expression of y as a rapidly increasing $\left(\frac{1}{m_{2j}^4},2j\right)$-s.c.c. First we notice that for $1\le k\le d_1$, $|z_k^*(y)|\le \frac{16}{m_{2t_k}m_{2j}}$ (Proposition 3.4).

Set now

$$I=\big\{k: k\in\{d_1+1,\dots,d\} \text{ and there exists } x_n \text{ with } \operatorname{supp} x_n\cap \operatorname{supp} z_k^*\neq\emptyset \text{ while } \operatorname{supp} x_n\cap\operatorname{supp} z_s^*=\emptyset \text{ for } s\neq k\big\}.$$

For $k\in I$ we set

$$T_k=\{n: n\in\{1,\dots,l\},\ \operatorname{supp} x_n\cap\operatorname{supp} z_k^*\neq\emptyset \text{ and } \operatorname{supp} x_n\cap\operatorname{supp} z_s^*=\emptyset \text{ for } s\neq k\}.$$

Now, for every $k\in I$

$$\left|\left(\sum_{r=d_1+1}^{d} z_r^*\right)\left(\sum_{n\in T_k} a_n x_n\right)\right| = \left|z_k^*\left(\sum_{n\in T_k} a_n x_n\right)\right|$$

and since $t_k>j$, by Proposition 3.4 we get (at worst) $\left|z_k^*\left(\sum_{n\in T_k} a_n x_n\right)\right|\le \frac{4}{m_{2t_k-1}}+4a_{n_k}$ for some $n_k\in T_k$. Observe that the set $\{n_k\}_{k\in I}$ satisfies the assumptions of Lemma 3.6, so $\sum_{k\in I} a_{n_k}<\frac{m_i}{m_{2j}^4}<\frac{1}{m_{2j}^3}$. We conclude that

$$\left|\left(\sum_{k=d_1+1}^{d} z_k^*\right)\left(\sum_{n\in\cup_{k\in I}T_k} a_n x_n\right)\right| = \left|\sum_{k\in I} z_k^*\left(\sum_{n\in T_k} a_n x_n\right)\right| \le \sum_{k=d_1+1}^{d}\frac{4}{m_{2t_k-1}}+\frac{1}{m_{2j}^3}.$$

Consider now $S=\{1,\dots,l\}\setminus\bigcup_{k\in I} T_k$. The set S satisfies again the assumptions of Lemma 3.6. On the other hand, since for every n, x_n is a $\left(\frac{1}{m_{2j_n}^4},2j_n\right)$-s.c.c. with $2j_n>2j+2>i$, for $n\in S$ we get by Proposition 3.3 that $\left|\left(\sum_{k=d_1+1}^{d} z_k^*\right)(a_n x_n)\right|<4a_n$. We conclude that

$$\left|\left(\sum_{k=d_1+1}^{d} z_k^*\right)\left(\sum_{n\in S} a_n x_n\right)\right| < 4\sum_{n\in S} a_n < \frac{4m_i}{m_{2j}^4}<\frac{1}{m_{2j}^3}.$$

This completes the proof.

3.8. Proposition. *Let j, $\{j_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^n$ be as in Proposition 3.5. Suppose that $2j+1<t_1<\dots<t_d$, $\{j_1,\dots,j_n\}\cap\{t_1,\dots,t_d\}=\emptyset$ and $\{z_s^*\}_{s=1}^d$ are successive and such that $z_s^*\in B_{2t_s}$ for $s=1,\dots,d$ and $\frac{1}{m_{2j+1}}(z_1^*+\dots+z_d^*)\in B_{2j+1}$. Then, for every set of scalars $(b_k)_{k=1}^n$,*

$$\left|\left(\sum_{s=1}^{d} z_s^*\right)\left(\sum_{k=1}^{n} b_k m_{2j_k} y_k\right)\right| < \sum_{k=1}^{n}|b_k|\frac{1}{m_{2j+2}^2}.$$

Proof. It follows from Lemma 3.7, using the lacunarity of the sequence $\{m_n\}_{n=0}^{\infty}$.

3.9. Proposition. *Let j, $\{j_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$, $\{\theta_k\}_{k=1}^n$ and $\{\epsilon_k\}_{k=1}^n$ be as in Proposition 3.5.*

For every ϕ in B_{2j+1} we have

$$\left|\phi\left(\sum_{k=1}^{n}\epsilon_k a_k m_{2j_k}\theta_k y_k\right)\right| \le \frac{1}{m_{2j+2}^2}.$$

Proof. Let $\phi = \frac{1}{m_{2j+1}}(x^*_{k_1-1}+y^*_{k_1}\cdots+y^*_{k_2}+z^*_{k_2+1}+\cdots+z^*_d)$, where $x^*_{k_1-1}=E_sy^*_{k_1-1}$ for some s and $z^*_{k_2+1}\neq y^*_{k_2+1}$, be the expression of ϕ as an element of B_{2j+1}.

We set for $k=1,\dots,n$, $z_k=\epsilon_k a_k m_{2j_k}\theta_k y_k$ and $w=\sum_{k=1}^n z_k$. Then

$$\begin{aligned}|(y^*_{k_1}+\cdots+y^*_{k_2})(w)| &= |(y^*_{k_1}(z_{k_1})+\cdots+y^*_{k_2}(z_{k_2})| \\ &= |a_{k_1}-a_{k_1+1}+\cdots+(-1)^{k_2-k_1}a_{k_2}|\leq a_{k_1}.\end{aligned}$$

Also, by Proposition 3.8,

$$\begin{aligned}|(z^*_{k_2+1}+\cdots+z^*_d)(w)| &\leq |z^*_{k_2+1}(z_{k_2+1})|+|(z^*_{k_2+2}+\cdots+z^*_d)(w)| \\ &\quad+\left|z^*_{k_2+1}\left(\sum_{k\neq k_2+1}z_k\right)\right|\leq 32a_{k_2+1}+\frac{8}{m^2_{2j+2}}.\end{aligned}$$

We conclude that

$$\begin{aligned}|\phi(w)| &\leq \frac{1}{m_{2j+1}}\left(|x^*_{k_1-1}(w)|+|(y^*_{k_1}+\cdots+y^*_{k_2})(w)|+|(z^*_{k_2+1}+\cdots+z^*_d)(w)|\right) \\ &\leq \frac{1}{m_{2j+1}}\left(32a_{k_1-1}+a_{k_1}+32a_{k_2}+\frac{8}{m^2_{2j+2}}\right)<\frac{1}{m^2_{2j+2}}.\end{aligned}$$

3.10. Proposition. *Let j, $\{j_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$ be as in Proposition 3.5.*

If $i>2j+1$ and ϕ is in B_i, we have the following: For every set of scalars $(b_k)_{k=1}^n$

$$\left|\phi\left(\sum_{k=1}^n b_k m_{2j_k}y_k\right)\right|\leq\begin{cases}\sum_{k=1}^n|b_k|\frac{16}{m_i} & \text{if } 2j+1<i<2j_1,\\ 16\sum_{k=1}^n\frac{|b_k|}{m_{2j+2}}+8|b_{k_0}| & \text{if } k_0 \text{ is maximal with } 2j_{k_0}\leq i.\end{cases}$$

Proof. The case $2j+1<i<2j_1$ follows from Proposition 3.4 (a), the case $2j_{k_0}\leq i<2j_{k_0+1}$ and $i>2j_n$ from Proposition 3.4 and the lacunarity of the sequence $\{m_n\}_{n=0}^\infty$.

3.11. Proposition. *Let j, $\{j_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$, $\{\theta_k\}_{k=1}^n$, $\{a_k\}_{k=1}^n$ and $\{\epsilon_k\}_{k=1}^n$ be as in Proposition 3.5.*

For every $i<2j+1$ and ϕ in B_i we have

$$\left|\phi\left(\sum_{k=1}^n\epsilon_k a_k m_{2j_k}\theta_k y_k\right)\right|\leq\frac{100}{m_{2j+2}}.$$

Proof. Let $\{K^s(\phi)\}_{s=0}^m$ be an analysis of the functional ϕ. We shall define a partition of $\{1,\dots,n\}$ into four sets, W, I_1, I_2 and I_3 and we shall consider the behaviour of ϕ on each one of the corresponding subsets of $\{y_k\}_{k=1}^n$ separately. We set

$$\begin{aligned}W=\Big\{k: k=1,\dots,n,\ &\text{there exists a functional } f \text{ in } \bigcup_{s=1}^m K^s(\phi) \text{ such that} \\ &\text{supp}\,\phi\cap\text{supp}\,y_k=\text{supp}\,f\cap\text{supp}\,y_k\neq\emptyset \text{ and } f\in B_{2j+1}\Big\}.\end{aligned}$$

For every k in W, we denote by f^k the functional which is of maximal support among the functionals $f\in\bigcup_{s=1}^m K^s(\phi)$ satisfying $\text{supp}\,\phi\cap\text{supp}\,y_k\neq\emptyset$ and $f\in B_{2j+1}$.

For $k\in W$, f^k is of the form $f^k=\frac{1}{m_{2j+1}}(x^*_{t_1}+y^*_{t_1^k+1}+\cdots+y^*_{t_2^k-1}+z^*_{t_2^k}+\cdots+z^*_d)$ where $x^*_{t_1^k}=E_l y^*_{t_1^k}$ for some $l\in\mathbf{N}$ and $z^*_{t_2^k}\neq y^*_{t_2^k}$.

We set

$$W_0 = \{k \in W : \; k > t_2^k\}.$$

Claim 1. $\left|\phi\left(\sum_{k\in W_0} \epsilon_k a_k m_{2j_k}\theta_k y_k\right)\right| \le \frac{4}{m_{2j+2}^2}.$

Proof. Let $k \in W_0$ and consider

$$f^k = \frac{1}{m_{2j+1}}(x_{t_1}^* + y_{t_1^k+1}^* + \cdots + y_{t_2^k-1}^* + z_{t_2^k}^* + \cdots + z_d^*).$$

Then, for $t \ge t_2^k$, we have $z_t^* \in B_{2s_t}$, where $2s_t = \Phi(y_1^*, \ldots , y_{t_2^k-1}^*, z_{t_2^k}^*, \ldots , z_{t-1}^*)$. On the other hand, $2j_k = \Phi(y_1^*, \ldots , y_{k-1}^*)$. Since $k > t_2^k$, we get $j_k \notin \{s_{t_2^k}, \ldots , s_d\}$. Then, by Proposition 3.8, $|f^k(m_{2j_k}y_k)| < \frac{1}{m_{2j+2}^2}$.

We conclude that $\left|\phi\left(\sum_{k\in W_0} \epsilon_k a_k m_{2j_k}\theta_k y_k\right)\right| \le \frac{4}{m_{2j+2}^2}$.

Now consider the set $W \setminus W_0$. The segments $\{[t_1^k, t_2^k] \cap W\}_{k\in W}$ define a partition of $W \setminus W_0$. We denote the mutually exclusive segments defined in this manner by $\{T_\sigma\}_{\sigma=1}^r$. We also set $k_\sigma = \min\{k : k \in T_\sigma\}$. Notice that

$$|f^{k_\sigma}(\sum_{k\in T_\sigma} \epsilon_k a_k m_{2j_k}\theta_k y_k)| \le 65a_{k_\sigma}.$$

We set now

$$W_1 = \bigcup \Big\{T_\sigma : \text{ for every } f \in \bigcup_{s=0}^{m} K^s(\phi) \text{ which strictly extends } f^{k_\sigma}, \text{ we have } f \in \bigcup_{q\le 2j} B_q\Big\}.$$

Claim 2. $\left|\phi\left(\sum_{k\in W_1} \epsilon_k a_k m_{2j_k}\theta_k y_k\right)\right| \le \frac{1}{m_{2j+2}^2}.$

Proof. Let $\sum_{k=1}^n a_k e_{p_k}$ be the $\left(\frac{1}{m_{2j+2}^4}, 2j+1\right)$-basic s.c.c. which defines the s.c.c. $\sum_{k=1}^n a_k y_k$. We show that there exists a functional ψ with $\|\psi\|_{2j}^* \le 1$ and such that

$$\left|\phi\left(\sum_{k\in W_1} \epsilon_k a_k m_{2j_k}\theta_k y_k\right)\right| \le 65\left|\psi\left(\sum_{\sigma=1}^{r} a_{k_\sigma} e_{p_{k_\sigma}}\right)\right| < \frac{1}{m_{2j+2}^2}.$$

We follow the same procedure as in the proof of Proposition 2.9. For every f in $\bigcup_{s=0}^m K^s(\phi)$ which extends some f^k, $k \in W_1$, we set

$$D_f = \{\sigma : \; k_\sigma \in W_1 \text{ and } f \text{ extends } f^{k_\sigma}\},$$

and define a functional g_f with $\|g_f\|_{2j}^* \le 1$ and such that

1) supp $g_f = \{p_{k_\sigma} : f \text{ extends } f^{k_\sigma}\}$,
2) $|f(\sum_{\sigma\in D_f}\sum_{k\in T_\sigma} \epsilon_k a_k m_{2j_k}\theta_k y_k)| \le 65g_f(\sum_{\sigma\in D_f} a_{k_\sigma} e_{p_{k_\sigma}})$.

The inductive construction is as follows:

Suppose that g_f has been defined for every $f \in K^{s-1}(\phi)$ which extends some f^k, $k \in W_1$. Let $f \in K^s(\phi)$. If $f = f^{k_\sigma}$ for some $k_\sigma \in W_1$ then we set $g_f = e_{p_{k_\sigma}}^*$. Then

$|f(\sum_{k\in T_\sigma}\epsilon_k a_k m_{2j_k}\theta_k y_k)| \le 65a_{k_\sigma} \le 65a_{k_\sigma} g_f(e_{p_{k_\sigma}})$. If f strictly extends some f^k, $k \in W_1$, then by the definition of W_1, f is of the form $\frac{1}{m_q}(f_1 + \cdots + f_d)$ where $q \le 2j$ and the set $\{\text{supp } f_1, \dots, \text{supp } f_d\}$ is $\mathcal{M}_q$-admissible. Then for $t = 1, \dots, d$ either f_t extends some f^k for $k \in W_1$ and the function g_{f_t} has already been defined or supp $f_t \cap$ supp $y_k = \emptyset$ for all $k \in W_1$. We set $I = \{t : t = 1, \dots, d$ and f_t extends f^k for some $k \in W_1\}$ and

$$g_f = \frac{1}{m_q}\sum_{t\in I} g_{f_t}.$$

It is easy to check that the set $\{\text{supp } g_{f_t} : t \in I\}$ is $\mathcal{M}_q$-admissible. Hence, by the inductive assumption we get $\|g_f\|_{2j}^* \le 1$.

Property (2) is also clear. This completes the proof of Claim 2.

We set $W_2 = W \setminus (W_0 \cup W_1)$.

Claim 3. $\left|\phi\left(\sum_{k\in W_2}\epsilon_k a_k m_{2j_k}\theta_k y_k\right)\right| \le \frac{32}{m_{2j+2}}.$

Proof. If $T_\sigma \subset W_2$ then there exists a function $f \in \bigcup_{s=0}^m K^s(\phi)$ strictly extending f^{k_σ} and such that $f \in B_q$ for some $q \ge 2j+2$. Then f is of the form $\frac{1}{m_q}(f_1 + \cdots f_d)$ where, for some $t = 1, \dots, d$, f_t extends f^{k_σ}. So $T_\sigma \subset$ supp f_t. So, for $k \in T_\sigma$,

$$|f(m_{2j_k}y_k)| = \frac{1}{m_q}|f_t(m_{2j_k}y_k)| \le \frac{8}{m_q} \le \frac{8}{m_{2j+2}}.$$

We conclude that $|\phi(\sum_{k\in W_2}\epsilon_k a_k m_{2j_k}\theta_k y_k)| \le \frac{8}{m_{2j+2}}$. This completes the proof of Claim 3.

Set now

$$I_1 = \Big\{k : k = 1, \dots, n,\ k \notin W \text{ and for every } f \in \bigcup_{s=0}^m K^s(\phi) \text{ such that}$$
$$\text{supp } f \cap \text{supp } y_k = \text{supp } \phi \cap \text{supp } y_k \ne \emptyset \text{ we have that } f \in \bigcup_{q\le 2j} B_q\Big\}.$$

Claim 4. $\left|\phi\left(\sum_{k\in I_1}\epsilon_k a_k m_{2j_k} y_k\right)\right| < \frac{1}{m_{2j+2}^2}.$

Proof. We prove that there exists a functional ψ such that $\|\psi\|_{2j}^* \le 1$ and

$$\left|\phi\left(\sum_{k\in I_1}\epsilon_k a_k m_{2j_k} y_k\right)\right| < 64\psi(\sum_{k\in I_1} a_k e_{p_k}).$$

This procedure is the same as in the proof of Proposition 2.9, using the fact that $|f(m_{2j_k}y_k)| < \frac{16}{m_q}$ for $f \in B_q$, $q < 2j+1$ (Proposition 3.4 (a)).

If now $k \in \{1, \dots, n\} \setminus (W \cup I_1)$ then there exists a functional $f \in \bigcup_{s=0}^m K^s(\phi)$ such that supp $\phi \cap$ supp $y_k =$ supp $f \cap$ supp $y_k \ne \emptyset$ and $f \in B_q$ for some $q \ge 2j+2$. Then by Proposition 3.10, $|f(m_{2j_k}y_k)| < \frac{16}{m_{2j+2}}$, unless $2j_k \le q < 2j_{k+1}$.

We set

$$I_2 = \Big\{k : k \notin W \bigcup I_1 \text{ and there exists } f \in \cup_{s=0}^{m} K^s(\phi) \text{ such that } \operatorname{supp} f \cap \operatorname{supp} y_k = \operatorname{supp} \phi \cap \operatorname{supp} y_k \neq \emptyset \text{and such that } |f(m_{2j_k} y_k)| \leq \frac{16}{m_{2j+2}}\Big\}$$

and

$$I_3 = \{1, \dots, n\} \setminus (W \cup I_1 \cup I_2).$$

It is easy to see that if $k \in I_3$ then the following holds:

There exists $f^k \in \bigcup_{s=0}^{m} K^s(\phi)$ with $\operatorname{supp} f \cap \operatorname{supp} y_k = \operatorname{supp} \phi \cap \operatorname{supp} y_k \neq \emptyset$ such that

(i) $f^k \in B_q$ where $2j_k \leq q < 2j_{k+1}$

(ii) For every $f \in \bigcup_{s=0}^{m} K^s(\phi)$ such that $\operatorname{supp} f \cap \operatorname{supp} y_k = \operatorname{supp} \phi \cap \operatorname{supp} y_k \neq \emptyset$ and $f \neq f^k$, $f \in \bigcup_{p \leq 2j} B_p$.

Note that if $k_1 \neq k_2$ belong to I_3 then $\operatorname{supp} f^{k_1} \cap \operatorname{supp} f^{k_2} = \emptyset$. This allows us to prove the following:

Claim 5. $\left|\phi\left(\sum_{k \in I_3} \epsilon_k a_k m_{2j_k} \theta_k y_k\right)\right| < 16 \left\|\sum_{k \in I_3} a_k e_{p_k}\right\|_{2j} < \frac{1}{m_{2j+2}}.$

Proof. The proof is again as in Proposition 2.9.

For $f \in \bigcup_{s=0}^{m} K^s(\phi)$ we set

$$D_f = \{k \in I_3 : \operatorname{supp} \phi \cap \operatorname{supp} y_k = \operatorname{supp} f \cap \operatorname{supp} y_k \neq \emptyset \text{ and } f \text{ extends } f^k\}$$

and we define g_f with $\operatorname{supp} g_f = \{p_k : k \in D_f\}$, $\|g_f\|_{2j}^* \leq 1$ and such that $|f(m_{2j_k} y_k)| \leq 16 g_f(e_{p_k})$ for every $k \in D_f$.

The inductive step is as follows:

If $f \in B_q$ for some $q \geq 2j+2$ then either $D_f = \emptyset$ in which case we set $g_f = 0$ or $f = f^k$ and $D_f = \{k\}$ for some $k \in I_3$. In the latter case we set $g_f = e_{p_k}^*$.

Suppose now that $f \in \bigcup_{p \leq 2j} B_p$, $f = \frac{1}{m_p}(f_1 + \cdots + f_d)$; if f extends f^k for some $k \in I_3$ then we set $g_f = \frac{1}{m_p}(g_{f_1} + \cdots + g_{f_d})$. Otherwise, we set $g_f = 0$. This completes the proof of the Claim 5.

By Claims 1, 2, 3, 4 and 5 we conclude that

$$\left|\phi\left(\sum_{k=1}^{n} \epsilon_k a_k m_{2j_k} \theta_k y_k\right)\right| \leq \frac{100}{m_{2j+2}}.$$

3.12. Proposition. *Let $(x_i)_{i \in \mathbf{N}}$, $(w_i)_{i \in \mathbf{N}}$ be two normalized block sequences in the space X. Then there exist $\{y_k\}_{k=1}^{n}$, $\{y_k^*\}_{k=1}^{n}$, $\{\theta_k\}_{k=1}^{n}$, $\{a_k\}_{k=1}^{n}$, satisfying the assumptions of Proposition 3.5 and such that, for k odd, y_k is a block of $(x_i)_{i \in \mathbf{N}}$ while, for k even, y_k is a block of $(w_i)_{i \in \mathbf{N}}$.*

Proof. Let j be given. We choose inductively a sequence $\{n_l\}_{l=0}^{\infty} \subset \mathbf{N}$, $(n_0 = 2)$, and vectors $u_{l,A} \in X$, $u_{l,A}^* \in X^*$, $l = 1, 2, \dots$, $A \subset \{1, \dots, l-1\}$ such that

(a) For every l and $A \subset \{1, \dots, l-1\}$, $u_{l,A}$ is a block of $(x_i)_{i \in \mathbf{N}}$ if $\#A$ is even and $u_{l,A}$ is a block of $(w_i)_{i \in \mathbf{N}}$ if $\#A$ is odd.

(b) For every $l = 1, 2, \dots$ and every $A \subset \{1, \dots, l-1\}$ the vectors $u_{l,A}$ and $u_{l,A}^*$ are supported inside $(n_{l-1}, n_l]$.

(c) Each $u_{l,A}$ is a $\left(\frac{1}{m_{2s}^4}, 2s\right)$-rapidly increasing s.c.c., $u_{l,A}^* \in B_{2s}$ and $u_{l,A}^*(u_{l,A}) \geq \frac{1}{4m_{2s}}$ where $s > 2j+1$ if $A = \emptyset$ and $2s = \Phi(u_{l_1,\emptyset}^*, u_{l_2,A_1}^*, \dots, u_{l_k,A_{k-1}}^*)$ if $A = \{l_1 < \cdots < l_k\}$ and $A_i = \{l_1, \dots, l_i\}$, $i = 1, 2, \dots, k-1$.

The inductive construction is straightforward.

Choose now $F \subset \{n_l\}_{l=1}^{\infty}$, $F = \{n_{l_1}, \dots, n_{l_k}\} \in \mathcal{F}_{s_{2j+1}}$ such that a convex combination $\sum_{n_l \in F} a_l e_{n_l}$ is a $\left(\frac{1}{m_{2j+2}^4}, 2j+1\right)$-basic s.c.c.

For $i = 1, \dots, k-1$, set $A_i = \{l_1, \dots, l_i\}$. Then it is easy to check that the sequence

$$u_{l_1,\emptyset}, u_{l_2,A_1}, \dots, u_{l_k,A_{k-1}}$$

and the corresponding one in X^* have the desired properties.

The following corollary is an immediate consequence of 3.5 and the previous proposition.

3.13. Corollary. *The Banach space X is Hereditarily Indecomposable. In particular X does not contain any unconditional basic sequence.*

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